

FREMLIN TENSOR PRODUCTS OF CONCAVIFICATIONS OF BANACH LATTICES

VLADIMIR G. TROITSKY AND OMID ZABETI

ABSTRACT. Suppose that E is a uniformly complete vector lattice and p_1, \dots, p_n are positive reals. We prove that the diagonal of the Fremlin projective tensor product of $E_{(p_1)}, \dots, E_{(p_n)}$ can be identified with $E_{(p)}$ where $p = p_1 + \dots + p_n$ and $E_{(p)}$ stands for the p -concavification of E . We also provide a variant of this result for Banach lattices. This extends the main result of [BBPTT].

1. INTRODUCTION AND MOTIVATION

We start with some motivation. Let E be a vector or a Banach lattice of functions on some set Ω , and consider a tensor product $E \tilde{\otimes} E$ of E with itself. It is often possible to view $E \tilde{\otimes} E$ as a space of functions on the square $\Omega \times \Omega$ with $(f \otimes g)(s, t) = f(s)g(t)$, where $f, g \in E$ and $s, t \in \Omega$. In particular, restricting this function to the diagonal $s = t$ gives just the product fg . Thus, the space of the restrictions of the elements of $E \tilde{\otimes} E$ to the diagonal can be identified with the space $\{fg : f, g \in E\}$, which is sometimes called the square of E , (see, e.g., [BvR01]). The concept of the square can be extended to uniformly complete vector lattices as the 2-concavification of E . Hence, one can expect that, for a uniformly complete vector lattice, the diagonal of an appropriate tensor product of E with itself can be identified with the 2-concavification of E . This was stated and proved formally in [BBPTT] for Fremlin projective tensor product of Banach lattices. It was shown there that the diagonal of the tensor product is lattice isometric to the 2-concavification of E ; the diagonal was defined as the quotient of the product over the ideal generated by all elementary tensors $x \otimes y$ with $x \perp y$.

In the present paper, we extend this result. Let us again provide some motivation. In the case when E is a space of functions on Ω , one can think of its p -concavification as $E_{(p)} = \{f^p : f \in E\}$ for $p > 0$. In this case, for positive real numbers p_1, \dots, p_n , the elementary tensors in $E_{(p_1)} \otimes \dots \otimes E_{(p_n)}$ can be thought of as functions on Ω^n of the form

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$f_1^{p_1} \otimes \cdots \otimes f_n^{p_n}(s_1, \dots, s_n) = f_1^{p_1}(s_1) \cdots f_n^{p_n}(s_n)$, where $f_1, \dots, f_n \in E$ and $s_1, \dots, s_n \in \Omega$. The restriction of this function to the diagonal is the product $f_1^{p_1} \cdots f_n^{p_n}$, which is an element of $E_{(p)}$ where $p = p_1 + \cdots + p_n$. That is, the diagonal of the tensor product of $E_{(p_1)}, \dots, E_{(p_n)}$ can be identified with $E_{(p)}$. In this paper, we formally state and prove this fact for the case when E is a uniformly complete vector lattice in Section 2 and when E is a Banach lattice in Section 3. In particular, this extends the result of [BBPTT] to vector lattices and to the product of an arbitrary number of copies of E (a variant of the latter statement was also independently obtained in [BB]).

2. PRODUCTS OF VECTOR LATTICES

Throughout this section, E will stand for a uniformly complete vector lattice. We need uniform completeness so that we can use positive homogeneous function calculus in E , see, e.g., Theorem 5 in [BvR01]. It is easy to see that every uniformly complete vector lattice is Archimedean.

Following [LT79, p. 53], by t^p , where $t \in \mathbb{R}$ and $p \in \mathbb{R}_+$, we mean $|t| \cdot \text{sign } t$; see also the discussion in Section 1.2 of [BBPTT]. In particular, if p_1, \dots, p_n are positive reals and $p = p_1 + \cdots + p_n$ then $|t|^{p_1}|t|^{p_2} \cdots |t|^{p_n} = |t|^p$ while $t^{p_1}|t|^{p_2} \cdots |t|^{p_n} = t^p$ for every $t \in \mathbb{R}$. It follows that $|x|^{p_1}|x|^{p_2} \cdots |x|^{p_n} = |x|^p$ and $x^{p_1}|x|^{p_2} \cdots |x|^{p_n} = x^p$ for every $x \in E$. In particular, $(x|x| \cdots |x|)^{\frac{1}{n}} = x$ for every $n \in \mathbb{N}$.

Suppose that p is a positive real number. Using function calculus, we can introduce new vector operations on E via $x \oplus y = (x^p + y^p)^{\frac{1}{p}}$ and $\alpha \odot x = \alpha^{\frac{1}{p}} x$, where $x, y \in E$ and $\alpha \in \mathbb{R}$. Together with these new operations and the original order and lattice structures, E becomes a vector lattice. This new vector lattice is denoted $E_{(p)}$ and called the ***p-concavification*** of E . It is easy to see that $E_{(p)}$ is still Archimedean.

We start by extending Theorem 1 and Corollary 2 in [BvR00]. Recall that an n -linear map φ from E^n to a vector lattice F is said to be **positive** if $\varphi(x_1, \dots, x_n) \geq 0$ whenever $x_1, \dots, x_n \geq 0$ and **orthosymmetric** if $\varphi(x_1, \dots, x_n) = 0$ whenever $|x_1| \wedge \cdots \wedge |x_n| = 0$; φ is said to be a **lattice n -morphism** if $|\varphi(x_1, \dots, x_n)| = \varphi(|x_1|, \dots, |x_n|)$ for any $x_1, \dots, x_n \in E$.

Theorem 1. *Suppose that $\varphi: C(K)^n \rightarrow F$, where K is a compact Hausdorff space, F is a vector lattice, $n \in \mathbb{N}$, and φ is an orthosymmetric positive n -linear map. Then $\varphi(x_1, \dots, x_n) = \varphi(x_1 \cdots x_n, 1, \dots, 1)$ for any $x_1, \dots, x_n \in C(K)$.*

Proof. The proof is by induction. The case $n = 1$ is trivial. The case $n = 2$ follows from Theorem 1 in [BvR00]. Suppose that $n > 2$ and the statement is true for

$n - 1$. Suppose that $\varphi: C(K)^n \rightarrow F$ is orthosymmetric positive and n -linear. Fix $0 \leq z \in C(K)$ and define $\varphi_z: C(K)^{n-1} \rightarrow F$ via $\varphi_z(x_1, \dots, x_{n-1}) = \varphi(x_1, \dots, x_{n-1}, z)$. Clearly, φ_z is orthosymmetric, positive, and $(n - 1)$ -linear. By the induction hypothesis, $\varphi_z(x_1, \dots, x_{n-1}) = \varphi_z(x_1 \cdots x_{n-1}, 1, \dots, 1)$ for all x_1, \dots, x_{n-1} in $C(K)$. It follows that

$$(1) \quad \varphi(x_1, \dots, x_n) = \varphi(x_1 \cdots x_{n-1}, 1, \dots, 1, x_n)$$

for all x_1, \dots, x_{n-1} and all $x_n > 0$. By linearity, (1) remains true for all $x_1, \dots, x_n \in C(K)$ as $x_n = x_n^+ - x_n^-$. Similarly, $\varphi(x_1, \dots, x_n) = \varphi(x_1 x_3 \cdots x_n, x_2, 1, \dots, 1)$ for all x_1, \dots, x_n in E . Applying (1) to the latter expression, we get $\varphi(x_1, \dots, x_n) = \varphi(x_1 \cdots x_n, 1, \dots, 1)$. This completes the induction. \square

Corollary 2. *Suppose that $\varphi: E^n \rightarrow F$, where E is a uniformly complete vector lattice, F is a vector lattice, $n \in \mathbb{N}$ and φ is an orthosymmetric positive n -linear map. Then $\varphi(x_1, \dots, x_n)$ is determined by $(x_1 \cdots x_n)^{\frac{1}{n}}$. Specifically,*

$$(2) \quad \varphi(x_1, \dots, x_n) = \varphi(x, |x|, \dots, |x|)$$

where $x = (x_1 \cdots x_n)^{\frac{1}{n}}$.

Proof. Suppose that $x_1, \dots, x_n \in E$. Let $e = |x_1| \vee \cdots \vee |x_n|$ and consider the principal ideal I_e . Then $x_1, \dots, x_n \in I_e$. Since I_e is lattice isomorphic to $C(K)$ for some compact Hausdorff space and the restriction of φ to $(I_e)^n$ is still orthosymmetric, positive, and n -linear, by the theorem we get (2). \square

Remark 3. The expression $\varphi(x, |x|, \dots, |x|)$ in (2) may look non-symmetric at the first glance. Lemma 2 may be restated in a more “symmetric” form as follows: $\varphi(x_1, \dots, x_n) = \varphi(x, \dots, x)$ for every positive x_1, \dots, x_n .

Next, we are going to generalize Corollary 2.

Theorem 4. *Suppose that $\varphi: E_{(p_1)} \times \cdots \times E_{(p_n)} \rightarrow F$, where E is a uniformly complete vector lattice, F is a vector lattice, $n \in \mathbb{N}$, p_1, \dots, p_n are positive reals, and φ is an orthosymmetric positive n -linear map. Then the following are true.*

- (i) *For all $x_1, \dots, x_n \in E$, we have $\varphi(x_1, \dots, x_n) = \varphi(x, |x|, \dots, |x|)$ where $x = x_1^{p_1/p} \cdots x_n^{p_n/p}$ with $p = p_1 + \cdots + p_n$.*
- (ii) *The map $\hat{\varphi}: E_{(p)} \rightarrow F$ defined by $\hat{\varphi}(x) = \varphi(x, |x|, \dots, |x|)$ is a positive linear map. If φ is a lattice n -morphism then $\hat{\varphi}$ is a lattice homomorphism.*

Proof. (i) First, we prove the statement for the case $E = C(K)$ for some Hausdorff compact space K . Define $\psi: E^n \rightarrow F$ via $\psi(x_1, \dots, x_n) = \varphi(x_1^{1/p_1}, \dots, x_n^{1/p_n})$. It is easy to see that ψ is an orthosymmetric positive n -linear map. Hence, applying Theorem 1 to ψ , we get

$$\begin{aligned} \varphi(x_1, \dots, x_n) &= \psi(x_1^{p_1}, \dots, x_n^{p_n}) = \psi(x_1^{p_1} \cdots x_n^{p_n}, 1, \dots, 1) \\ &= \psi(x^p, 1, \dots, 1) = \psi(x^{p_1}, |x|^{p_2}, \dots, |x|^{p_n}) = \varphi(x, |x|, \dots, |x|). \end{aligned}$$

Now suppose that E is a uniformly complete vector lattice. Choose $e \in E_+$ such that $x_1, \dots, x_n \in I_e$. Recall that I_e is lattice isomorphic to $C(K)$ for some Hausdorff compact space K . It is easy to see that $(I_e)_{(p_i)}$ is an ideal in $E_{(p_i)}$. The restriction of φ to $(I_e)_{(p_1)} \times \cdots \times (I_e)_{(p_n)}$ is again an orthosymmetric positive n -linear map, so the conclusion follows from the first part of the proof.

(ii) The proof that $\hat{\varphi}(\alpha \odot x) = \alpha \hat{\varphi}(x)$ is straightforward. We proceed to check additivity. Again, suppose first that $E = C(K)$ for some compact Hausdorff space K ; let ψ be as before. Take any $x, y \in E$ and put $z = x \oplus y$ in $E_{(p)}$, i.e., $z = (x^p + y^p)^{1/p}$. Then, again applying Theorem 1 to ψ , we have

$$\begin{aligned} \varphi(z, |z|, \dots, |z|) &= \psi(z^{p_1}, |z|^{p_2}, \dots, |z|^{p_n}) = \psi(z^p, 1, \dots, 1) \\ &= \psi(x^p, 1, \dots, 1) + \psi(y^p, 1, \dots, 1) = \varphi(x, |x|, \dots, |x|) + \varphi(y, |y|, \dots, |y|). \end{aligned}$$

Hence,

$$(3) \quad \varphi((x^p + y^p)^{\frac{1}{p}}, |x^p + y^p|^{\frac{1}{p}}, \dots, |x^p + y^p|^{\frac{1}{p}}) = \varphi(x, |x|, \dots, |x|) + \varphi(y, |y|, \dots, |y|)$$

Now suppose that E is an arbitrary uniformly complete vector lattice and $x, y \in E$. Taking $e = |x| \vee |y|$ and proceeding as in (i), one can see that (3) still holds, which yields $\hat{\varphi}(x \oplus y) = \hat{\varphi}(x) + \hat{\varphi}(y)$. \square

Corollary 5. *Let E be a uniformly complete vector lattice and p_1, \dots, p_n positive reals; put $p = p_1 + \cdots + p_n$. For $x_1, \dots, x_n \in E$, define $\mu(x_1, \dots, x_n) = x_1^{p_1/p} \cdots x_n^{p_n/p}$. Then*

- (i) $\mu: E_{(p_1)} \times \cdots \times E_{(p_n)} \rightarrow E_{(p)}$ is an orthosymmetric lattice n -morphism;
- (ii) For every vector lattice F there is a one to one correspondence between orthosymmetric positive n -linear maps $\varphi: E_{(p_1)} \times \cdots \times E_{(p_n)} \rightarrow F$ and positive linear maps $T: E_{(p)} \rightarrow F$ such that $\varphi = T\mu$ and $Tx = \varphi(x, |x|, \dots, |x|)$. Moreover, φ is a lattice n -morphism iff T is a lattice homomorphism.

Proof. (i) is straightforward. Note that $\mu(x, |x|, \dots, |x|) = x$ for every $x \in E$.

FIGURE 1.

$$\begin{array}{ccc}
E_{(p_1)} \times \cdots \times E_{(p_n)} & \xrightarrow{\varphi} & F \\
\mu \downarrow & \nearrow T & \\
E_{(p)} & &
\end{array}$$

(ii) If $T: E_{(p)} \rightarrow F$ is a positive linear map then setting $\varphi := T\mu$ defines an orthosymmetric positive n -linear map on $E_{(p_1)} \times \cdots \times E_{(p_n)}$ and

$$\varphi(x, |x|, \dots, |x|) = T\mu(x, |x|, \dots, |x|) = Tx.$$

Conversely, suppose that $\varphi: E_{(p_1)} \times \cdots \times E_{(p_n)} \rightarrow F$ is an orthosymmetric positive n -linear map; define $T: E_{(p)} \rightarrow F$ via $Tx := \varphi(x, |x|, \dots, |x|)$. Then T is a positive linear operator by Theorem 4(ii). Given $x_1, \dots, x_n \in E$, put $x = \mu(x_1, \dots, x_n)$. It follows from Theorem 4(i) that

$$T\mu(x_1, \dots, x_n) = Tx = \varphi(x, |x|, \dots, |x|) = \varphi(x_1, \dots, x_n),$$

so that $T\mu = \varphi$. □

We will use the fact, due to Luxemburg and Moore, that if J is an ideal in a vector lattice F then the quotient vector lattice F/J is Archimedean iff J is uniformly closed, see, e.g., Theorem 2.23 in [AB06] and the discussion preceding it. Recall that given a set A in a vector lattice F , A is uniformly closed if the limit of every uniformly convergent net in A is contained in A (it is easy to see that it suffices to consider sequences). The uniform closure of a set A in F is the set of the uniform limits of sequences in A ; it can be easily verified that this set is uniformly closed. Clearly, the uniform closure of an ideal is an ideal. Hence, for every set A , the uniform closure of the ideal generated by A is the smallest uniformly closed ideal containing A .

For Archimedean vector lattices E_1, \dots, E_n , we write $E_1 \bar{\otimes} \dots \bar{\otimes} E_n$ for their Fremlin vector lattice tensor product; see [Frem72, Frem74].

Theorem 6. *Let E be a uniformly complete vector lattice, p_1, \dots, p_n positive reals, and I_o the uniformly closed ideal in $E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)}$ generated by the elementary tensors of form $x_1 \otimes \cdots \otimes x_n$ with $\bigwedge_{i=1}^n |x_i| = 0$. Then the quotient $(E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)})/I_o$ is lattice isomorphic to $E_{(p)}$.*

Proof. Consider the diagram

$$(4) \quad E_{(p_1)} \times \cdots \times E_{(p_n)} \xrightarrow{\otimes} E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)} \xrightarrow{q} (E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)})/I_o$$

where q is the quotient map; $q(u) = u + I_o =: \tilde{u}$ for $u \in E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)}$.

Let μ be as in Corollary 5. By the universal property of the tensor product (see, e.g., [Frem72, Theorem 4.2(i)]), there exists a lattice homomorphism $M: E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)} \rightarrow E_{(p)}$ such that $M(x_1 \otimes \dots \otimes x_n) = \mu(x_1, \dots, x_n)$ for all x_1, \dots, x_n . Since μ is orthosymmetric, $M(x_1 \otimes \dots \otimes x_n) = 0$ whenever $\bigwedge_{i=1}^n |x_i| = 0$. Since M is a lattice homomorphism, it follows that M vanishes on I_o . Therefore, the quotient operator \widetilde{M} is well defined: for $u \in E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)}$ we have $\widetilde{M}\tilde{u} = Mu$. Furthermore, since q is a lattice homomorphism (see, e.g., [AB06, Theorem 2.22]), it is easy to see that \widetilde{M} is a lattice homomorphism as well.

Note that M is onto because for every $x \in E_{(p)}$ we have $x = M(x \otimes |x| \otimes \dots \otimes |x|)$. It follows that \widetilde{M} is onto. It is left to show that \widetilde{M} is one-to-one.

The composition map $q \otimes$ in (4) is an orthosymmetric lattice n -morphism. By Corollary 5, there is a lattice homomorphism $T: E_{(p)} \rightarrow (E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)})/I_o$ such that $q \otimes = T\mu$ and

$$Tx = (q \otimes)(x, |x|, \dots, |x|) = x \otimes |x| \otimes \dots \otimes |x| + I_o$$

for every $x \in E_{(p)}$. It follows that for every x_1, \dots, x_n we have

$$\begin{aligned} T\widetilde{M}(x_1 \otimes \dots \otimes x_n + I_o) &= TM(x_1 \otimes \dots \otimes x_n) \\ &= T\mu(x_1, \dots, x_n) = (q \otimes)(x_1, \dots, x_n) = x_1 \otimes \dots \otimes x_n + I_o, \end{aligned}$$

so that $T\widetilde{M}$ is the identity on the quotient of algebraic tensor product $(E_{(p_1)} \otimes \dots \otimes E_{(p_n)})/I_o$. We claim that it is still the identity map on $(E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)})/I_o$; this would imply that \widetilde{M} is one-to-one and complete the proof.

Suppose that $u \in E_{(p_1)} \bar{\otimes} \dots \bar{\otimes} E_{(p_n)}$. By [Frem72, Theorem 4.2(i)], there exist $w := z_1 \otimes \dots \otimes z_n$ in $E_{(p_1)} \otimes \dots \otimes E_{(p_n)}$ with $z_1, \dots, z_n \geq 0$ such that for every positive real δ there exists $v \in E_{(p_1)} \otimes \dots \otimes E_{(p_n)}$ with $|u - v| \leq \delta w$. Since the quotient map q is a lattice homomorphism, we get $|\tilde{u} - \tilde{v}| \leq \delta \tilde{w}$. Since T and \widetilde{M} are lattice homomorphisms and, by the preceding paragraph, $T\widetilde{M}$ preserves \tilde{v} and \tilde{z} , we get $|T\widetilde{M}\tilde{u} - \tilde{v}| \leq \delta \tilde{w}$. It follows that $|T\widetilde{M}\tilde{u} - \tilde{u}| \leq |T\widetilde{M}\tilde{u} - \tilde{v}| + |\tilde{u} - \tilde{v}| \leq 2\delta \tilde{w}$. Since δ is arbitrary, it follows by the Archimedean property that $T\widetilde{M}\tilde{u} = \tilde{u}$. \square

Remark 7. Note that the lattice isomorphism constructed in the proof of the theorem sends $x_1 \otimes \dots \otimes x_n + I_o$ into $x_1^{p_1/p} \dots x_n^{p_n/p}$, while its inverse sends x to $x \otimes |x| \otimes \dots \otimes |x| + I_o$ for every x .

3. PRODUCTS OF BANACH LATTICES

Now suppose that E is a Banach lattice and p is a positive real number. For each $x \in E_{(p)}$ we define

$$\|x\|_{(p)} = \inf \left\{ \sum_{i=1}^k \|v_i\|^p : |x| \leq v_1 \oplus \cdots \oplus v_k, v_1, \dots, v_k \geq 0 \right\}.$$

It is easy to see that this is a lattice seminorm on $E_{(p)}$. We will write $x \sim y$ if the difference $x \ominus y$ is in the kernel of this seminorm. For $x \in E$ we will write $[x]$ for the equivalence class of x . Let $E_{[p]}$ be the completion of $E_{(p)}/\ker \|\cdot\|_{(p)}$. Then $E_{[p]}$ is a Banach lattice.

Let's compare this definition with the concepts of the p -convexification and the p -conconcavification of a Banach lattice, e.g., in [LT79]. If $p > 1$ and E is p -convex then $\|\cdot\|_{(p)}$ is a complete norm on $E_{(p)}$, hence $E_{[p]} = E_{(p)}$, and this is exactly the p -conconcavification of E in the sense of [LT79]. In particular, if E is p -convex with constant 1 then $\|\cdot\|^p$ is already a norm, so that, by the triangle inequality, we have $\|\cdot\|_{(p)} = \|\cdot\|^p$. On the other hand, let $0 < p < 1$. Put $q = \frac{1}{p} > 1$. As in the construction of the q -convexification $E^{(q)}$ of E in [LT79], we see that $\|\cdot\|^p$ is already a norm on $E_{(p)}$, so that $\|\cdot\|_{(p)} = \|\cdot\|^p$. In this case, $E_{[p]} = E_{(p)} = E^{(q)}$. Thus, the $E_{[p]}$ notation allows us to unify convexifications and concavifications, and it does not make any assumptions on E besides being a Banach (or even a normed) lattice.

If E_1, \dots, E_n are Banach lattices, we write $E_1 \otimes_{\pi} \dots \otimes_{\pi} E_n$ for the Fremlin projective tensor of E_1, \dots, E_n as in [Frem74]; we denote the norm on this product by $\|\cdot\|_{\pi}$. We will make use of the following universal property of this tensor product, which is essentially Theorem 1E(iii,iv) in [Frem74] (see also Part (d) of Section 2 in [Sch84]).

Lemma 8. *Suppose E_1, \dots, E_n and F are Banach lattices. There is an one-to-one norm preserving correspondence between continuous positive n -linear maps $\varphi : E_1 \times \dots \times E_n \rightarrow F$ and positive operators $\varphi^{\otimes} : E_1 \otimes_{\pi} \dots \otimes_{\pi} E_n \rightarrow F$ such that $\varphi(x_1, \dots, x_n) = \varphi^{\otimes}(x_1 \otimes \dots \otimes x_n)$. Moreover, φ^{\otimes} is a lattice homomorphism if and only if φ is a lattice n -morphism.*

Lemma 9. *Let E be a Banach lattice and μ be as in Corollary 5. Then $\|\mu\| \leq 1$.*

Proof. By Proposition 1.d.2(i) of [LT79], we have

$$(5) \quad \|\mu(x_1, \dots, x_n)\| \leq \|x_1\|^{\frac{p_1}{p}} \cdots \|x_n\|^{\frac{p_n}{p}}$$

for every x_1, \dots, x_n . Fix $x_1, \dots, x_n \in E$. As in the definition of $\|\cdot\|_{(p)}$, suppose that

$$(6) \quad |x_1| \leq v_1^{(1)} \oplus \dots \oplus v_{k_1}^{(1)}, \quad \dots, \quad |x_n| \leq v_1^{(n)} \oplus \dots \oplus v_{k_n}^{(n)}$$

for some positive $v_i^{(m)}$'s. Since μ is a lattice n -morphism, we have

$$|\mu(x_1, \dots, x_n)| = \mu(|x_1|, \dots, |x_n|) \leq \mu\left(\bigoplus_{i_1=1}^{k_1} v_{i_1}^{(1)}, \dots, \bigoplus_{i_n=1}^{k_n} v_{i_n}^{(n)}\right) = \bigoplus_{i_1, \dots, i_n} \mu(v_{i_1}^{(1)}, \dots, v_{i_n}^{(n)})$$

where each i_m runs from 1 to k_m . The definition of $\|\cdot\|_{(p)}$ yields

$$\|\mu(x_1, \dots, x_n)\|_{(p)} \leq \sum_{i_1, \dots, i_n} \|\mu(v_{i_1}^{(1)}, \dots, v_{i_n}^{(n)})\|^p.$$

It follows from (5) that

$$\|\mu(x_1, \dots, x_n)\|_{(p)} \leq \sum_{i_1, \dots, i_n} \|v_{i_1}^{(1)}\|^{p_1} \dots \|v_{i_n}^{(n)}\|^{p_n} = \left(\sum_{i_1=1}^{k_1} \|v_{i_1}^{(1)}\|^{p_1}\right) \dots \left(\sum_{i_n=1}^{k_n} \|v_{i_n}^{(n)}\|^{p_n}\right).$$

Taking infimum over all positive $v_i^{(m)}$'s in (6), we get

$$\|\mu(x_1, \dots, x_n)\|_{(p)} \leq \|x_1\|_{(p_1)} \dots \|x_n\|_{(p_n)}.$$

□

Theorem 10. *Let E be a Banach lattice and p_1, \dots, p_n positive reals. Put $F = E_{[p_1]} \otimes_{[p]} \dots \otimes_{[p]} E_{[p_n]}$. Let I_{oc} be the norm closed ideal in F generated by elementary tensors $[x_1] \otimes \dots \otimes [x_n]$ with $\bigwedge_{i=1}^n |x_i| = 0$. Then F/I_{oc} is lattice isometric to $E_{[p]}$ where $p = p_1 + \dots + p_n$.*

Proof. Let μ be as in Corollary 5. Fix x_1, \dots, x_n in E . Take any x'_1, \dots, x'_n in E such that $\|x'_i - x_i\|_{(p_i)} = 0$ as $i = 1, \dots, n$. Then it follows from Lemma 9 that

$$\|\mu(x_1, x_2, \dots, x_n) \ominus \mu(x'_1, x_2, \dots, x_n)\|_{(p)} = \|\mu(x_1 \ominus x'_1, x_2, \dots, x_n)\|_{(p)} = 0,$$

so that $\mu(x_1, x_2, \dots, x_n) \sim \mu(x'_1, x_2, \dots, x_n)$ in $E_{(p)}$. Iterating this process, we see that $\mu(x'_1, \dots, x'_n) \sim \mu(x_1, \dots, x_n)$. It follows that μ induces a map

$$\tilde{\mu}: (E_{(p_1)}/\ker\|\cdot\|_{(p_1)}) \times \dots \times (E_{(p_n)}/\ker\|\cdot\|_{(p_n)}) \rightarrow E_{(p)}/\ker\|\cdot\|_{(p)}$$

via $\tilde{\mu}([x_1], \dots, [x_n]) = [\mu(x_1, \dots, x_n)]$. Lemma 9 implies that $\|\tilde{\mu}\| \leq 1$, so that it extends by continuity to a map $\varphi: E_{[p_1]} \times \dots \times E_{[p_n]} \rightarrow E_{[p]}$. It is easy to see that φ is still an orthosymmetric lattice n -morphism and $\|\varphi\| \leq 1$. As in Lemma 8, φ gives rise to a lattice homomorphism $\varphi^\otimes: F \rightarrow E_{[p]}$ such that $\|\varphi^\otimes\| \leq 1$ and

$$(7) \quad \varphi^\otimes([x_1] \otimes \dots \otimes [x_n]) = \varphi([x_1], \dots, [x_n]) = [\mu(x_1, \dots, x_n)].$$

The latter implies that φ^\otimes vanishes on I_{oc} . This, in turn, implies that φ^\otimes induces a map $\widetilde{\varphi^\otimes}: F/I_{oc} \rightarrow E_{[p]}$, which is again a lattice homomorphism and $\|\widetilde{\varphi^\otimes}\| \leq 1$.

Consider the map $\psi: E_{(p_1)} \times \cdots \times E_{(p_n)} \rightarrow F/I_{oc}$ defined by $\psi(x_1, \dots, x_n) = [x_1] \otimes \cdots \otimes [x_n] + I_{oc}$. It can be easily verified that ψ is an orthosymmetric lattice n -morphism. It follows from Corollary 5 that there exists a lattice homomorphism $T: E_{(p)} \rightarrow F/I_{oc}$ such that $\psi = T\mu$ and

$$Tx = \psi(x, |x|, \dots, |x|) = [x] \otimes [|x|] \otimes \cdots \otimes [|x|] + I_{oc}$$

for every $x \in E_{(p)}$.

We claim that $\|Tx\| \leq \|x\|_{(p)}$. Note first that as $\|\cdot\|_{[p]}$ is a cross-norm, we have

$$(8) \quad \|Tx\| \leq \|[x]\|_{E_{[p_1]}} \cdots \|[x]\|_{E_{[p_n]}} \leq \|x\|_{(p_1)} \cdots \|x\|_{(p_n)} \leq \|x\|^{p_1} \cdots \|x\|^{p_n} = \|x\|^p.$$

Suppose that $|x| \leq v_1 \oplus \cdots \oplus v_m$ for some positive v_1, \dots, v_m , as in the definition of $\|\cdot\|_{(p)}$. Then $|Tx| = T|x| \leq \sum_{i=1}^m Tv_i$, so that $\|Tx\| \leq \sum_{i=1}^m \|Tv_i\| \leq \sum_{i=1}^m \|v_i\|^p$ by (8). It follows that $\|Tx\| \leq \|x\|_{(p)}$.

Therefore, T induces an operator from $E_{(p)}/\ker\|\cdot\|_{(p)}$ to I_{oc} and, furthermore, an operator from $E_{[p]}$ to F/I_{oc} , which we will denote \widetilde{T} , such that $\widetilde{T}[x] = Tx$ for every $x \in E_{(p)}$. Clearly, \widetilde{T} is still a lattice homomorphism and $\|\widetilde{T}\| \leq 1$. We will show that \widetilde{T} is the inverse of $\widetilde{\varphi^\otimes}$. This will complete the proof because this would imply that $\widetilde{\varphi^\otimes}$ is a surjective lattice isomorphism; it would follow from $\|\widetilde{\varphi^\otimes}\| \leq 1$ and $\|\widetilde{T}\| \leq 1$ that $\widetilde{\varphi^\otimes}$ is an isometry.

Take any $x \in E$ and consider the corresponding class $[x]$ in $E_{[p]}$. Using (7), we get

$$\widetilde{\varphi^\otimes}\widetilde{T}[x] = \widetilde{\varphi^\otimes}Tx = \varphi^\otimes([x] \otimes [|x|] \otimes \cdots \otimes [|x|]) = [\mu(x, |x|, \dots, |x|)] = [x].$$

Therefore, $\widetilde{\varphi^\otimes}\widetilde{T}$ is the identity on $E_{[p]}$. Conversely, for any x_1, \dots, x_n in E it follows by (7) that

$$\begin{aligned} \widetilde{T}\widetilde{\varphi^\otimes}([x_1] \otimes \cdots \otimes [x_n] + I_{oc}) &= \widetilde{T}[\mu(x_1, \dots, x_n)] = T\mu(x_1, \dots, x_n) \\ &= \psi(x_1, \dots, x_n) = [x_1] \otimes \cdots \otimes [x_n] + I_{oc}. \end{aligned}$$

Therefore, $\widetilde{T}\widetilde{\varphi^\otimes}$ is the identity on the linear subspace of F/I_{oc} that corresponds to the algebraic tensor product, i.e., on $q(E_{[p_1]} \otimes \cdots \otimes E_{[p_n]})$, where q is the canonical quotient map from F to F/I_{oc} . Since $E_{[p_1]} \otimes \cdots \otimes E_{[p_n]}$ is dense in F , it follows that q maps it into a dense subspace of F/I_{oc} . Therefore, $\widetilde{T}\widetilde{\varphi^\otimes}$ is the identity on a dense subspace of F/I_{oc} , hence on all of F/I_{oc} . \square

Remark 11. Note that the isometry from F/I_{oc} onto $E_{[p]}$ constructed in the proof of Theorem 10 sends $[x_1] \otimes \cdots \otimes [x_n] + I_{\text{oc}}$ into $[x_1^{p_1/p} \cdots x_n^{p_n/p}]$, while its inverse sends $[x]$ to $[x] \otimes [|x|] \otimes \cdots \otimes [|x|] + I_{\text{oc}}$ for every x .

Applying the theorem with $p_1 = \cdots = p_n = 1$, we obtain the following corollary, which extends the main result of [BBPTT]; see also [BB].

Corollary 12. *Suppose that E is a Banach lattice. Let I_{oc} be the closed ideal in $E \otimes_{[\cdot]} \cdots \otimes_{[\cdot]} E$ generated by the elementary tensors $x_1 \otimes \cdots \otimes x_n$ where $\bigwedge_{i=1}^n |x_i| = 0$. Then $(E \otimes_{[\cdot]} \cdots \otimes_{[\cdot]} E)/I_{\text{oc}}$ is lattice isometric to $E_{[n]}$.*

Recall that if $p < 1$ then $E_{[p]} = E^{(q)}$, the q -convexification of E where $q = \frac{1}{p}$. Hence, putting $q_i = \frac{1}{p_i}$ in the theorem, we obtain the following.

Corollary 13. *Suppose that E is a Banach lattice q_1, \dots, q_n are positive reals such that their geometric mean $q := (\frac{1}{q_1} + \cdots + \frac{1}{q_n})^{-1}$ satisfies $q \geq 1$. Let I_{oc} be the closed ideal in $E^{(q_1)} \otimes_{[\cdot]} \cdots \otimes_{[\cdot]} E^{(q_n)}$ generated by the elementary tensors $x_1 \otimes \cdots \otimes x_n$ where $\bigwedge_{i=1}^n |x_i| = 0$. Then $(E^{(q_1)} \otimes_{[\cdot]} \cdots \otimes_{[\cdot]} E^{(q_n)})/I_{\text{oc}}$ is lattice isometric to $E^{(q)}$.*

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(V.G. Troitsky) DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, AB, T6G 2G1. CANADA
E-mail address: troitsky@ualberta.ca

(O. Zabeti) DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P.O. BOX 1159, MASHHAD 91775, IRAN
E-mail address: ozabeti@yahoo.ca